1. Let X be a first-countable topological space and let $A \subseteq X$.

(a) Prove (reprove rather) that for any $x \in \overline{A}$, there is a sequence in A converging to x.

(b) Conclude that if A is dense, then for every $x \in X$, there is a sequence in A converging to x.

(c) Deduce that for every real $r \in \mathbb{R}$, there are sequences $(q_n) \subseteq \mathbb{Q}$ and $(r_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ converging to r.

2. Consider $\mathbb{R}$ with its standard metric.

(a) Prove that the characteristic/indicator function $\mathbb{1}_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ is discontinuous at every point $x_0 \in \mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows: $f|_{\mathbb{R}\setminus\mathbb{Q}} := 0$ and for each $q \in \mathbb{Q}$, $f(q) = \frac{1}{n}$, where $q = \frac{m}{n}$ is irreducible and $m \in \mathbb{Z}$, $n \in \mathbb{N}^+$. Prove that f is continuous at every irrational and it is discontinuous at every rational.

HINT: Recall that in metric spaces, continuity and sequential continuity are equivalent. Use the fact that for every real, there is a sequence of rationals converging to it and their denominators necessarily grow.

3. Let X, Y be topological spaces, where Y is T_1, i.e., for any distinct points $y_0, y_1 \in Y$, there is an open set $U_0 \ni y_0$ that does not contain y_1[1]. Prove that a function $f : X \to Y$ is sequentially continuous (i.e., sequentially continuous at every point in X) if and only if f maps convergent sequences in X to convergent sequences in Y.

4. Let X be a first-countable topological space and let (x_n) be a sequence in it. Let T_n be the n^{th} tail set of the sequence, i.e,

$$T_n := \{x_m : m \geqslant n\}.$$

Prove that for each $x \in \bigcap_{n\in\mathbb{N}} \overline{T_n}$, there is a subsequence[2] $(x_{n_k})_{k\in\mathbb{N}}$ converging to x.

5. Let (X, d) be a metric space. Show that every Cauchy sequence (x_n) admits a subsequence (x_{n_k}) with the property that $d(x_{n_k}, x_{n_\ell}) \leqslant 2^{-k}$ for all $k \leqslant \ell$.

6. Recall the following characterization of completeness for metric spaces.

Theorem (Completeness via nonempty intersections)**.** *For a metric space (X, d), the following are equivalent.*

(1) *(X, d) is complete.*

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[1]This is a weaker property than Hausdorff.

[2]Formally speaking, a *subsequence* of a sequence (x_n) is a sequence (y_k) such that there is a strictly increasing map $\mathbb{N} \to \mathbb{N}$ denoted $k \mapsto n_k$ such that $y_k = x_{n_k}$ for each $k \in \mathbb{N}$. Instead of writing (y_k), we simply write (x_{n_k}), with the understanding that the n_k are increasing, i.e., $n_0 < n_1 < n_2 < \dots$

(2) *Every decreasing*[3] *sequence* (C_n) *of closed sets with vanishing diameter*[4] *has a nonempty intersection.*

(3) *Every decreasing sequence* (B_n) *of closed balls of vanishing radius (equivalently, vanishing diameter) has a nonempty intersection.*

In class we proved (1)⇔(2). Noting that (2)⇒(3) is trivial, prove (3)⇒(1) following the steps below.

(i) Assume (3), and to prove (1), fix a Cauchy sequence (x_n). Aiming to prove that it converges, argue that it is enough to show that a subsequence of (x_n) converges.

(ii) Let $(y_k) := (x_{n_k})$ be a subsequence as in Problem 5.

(iii) Prove that for each $k \in \mathbb{N}$, the closed ball $\bar{B}(y_k, 2^{-k})$ contains the k^{th} tail set $T_k := \{y_m : m \geqslant k\}$ and $\bar{B}(y_k, 2 \cdot 2^{-k}) \supseteq \bar{B}(y_{k+1}, 2 \cdot 2^{-(k+1)})$.

(iv) Conclude that $\bigcap_{k \in \mathbb{N}} \bar{B}(y_k, 2 \cdot 2^{-k}) \neq \emptyset$, so $\bigcap_{k \in \mathbb{N}} \bar{B}(y_k, 2 \cdot 2^{-k}) = \{x\}$ for some x.

(v) Prove that there is a subsequence (y_{k_m}) converging to x, thus showing that a subsequence of (x_n) converges.

7. Let $\mathcal{T}$ be the (weird) topology on $\mathbb{R}$ generated by the sets $\{0, r\}$, where r ranges over $\mathbb{R}$. In particular $\{0, 0\} = \{0\}$ is open.

(a) Draw this topological space as a flower (daisy) with 0 being the center and the sets $\{0, r\}$, with $r \neq 0$, being the petals.

(b) Observe that the sets $\{0, r\}$, with $r \in \mathbb{R}$, actually form a base for the topology.

(c) Prove that this space is not Hausdorff; in fact, $\{0\}$ is not closed.

(d) Show that for each $r \in \mathbb{R}$, the collection $\big\{\{0, r\}\big\}$ is a neighborhood base for r. Conclude that $\mathcal{T}$ is first-countable.

(e) Yet, show that $\mathcal{T}$ is not second-countable.

(f) Prove that the set $\{0\}$ is dense. Conclude that this space is separable.

8. Reprove carefully that for a first-countable space, compactness implies sequential compactness.

9. Let (X, d) be a metric space. For $\varepsilon > 0$, call a set $F \subseteq X$ an *ε-net* if the open balls $B(x, \varepsilon)$ with $x \in F$ cover all of X, i.e., $X = \bigcup_{x \in F} B(x, \varepsilon)$. Call the space X *totally bounded* if for each $\varepsilon > 0$, there is a finite ε-net. Prove that (X, d) is sequentially compact if and only if it is complete and totally bounded.

Hint: Prove ⇒ via the contrapositive. The proof of ⇐ is very similar to that of the Bolzano–Weierstrass theorem, except that in the latter we always put infinitely-many pigeons into two holes, whereas here, we put infinitely-many pigeons into finitely-many holes (the ε-nets are finite).

[3] A sequence (S_n) is said to be *decreasing* (resp., *increasing*) if $C_n \supseteq C_{n+1}$ (resp., $C_n \subseteq C_{n+1}$) for all $n \in \mathbb{N}$.

[4] *Vanishing diameter* means that $\operatorname{diam}(C_n) \to 0$, where for $A \subseteq X$, $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$.

Remark: By Problem 8, compactness implies sequential compactness for metric spaces. The converse is also true, but it's a bit harder to show. Ask me if you are curious.

10. Consider $\mathbb{R}$ with its standard metric.

(a) Prove that a subset of $\mathbb{R}$ is complete (as a sub-metric space of $\mathbb{R}$) if and only if it is closed.

(b) Prove that a subset of $\mathbb{R}$ is totally bounded if and only if it is bounded.

(c) Conclude that a subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.